BROWNIAN MOTION DEVELOPMENT FOR MONTE CARLO METHOD APPLIED ON EUROPEAN STYLE OPTION PRICE FORECASTING

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Abstract: *In forecasting values of random series (values of options and stocks in the future) Brownian Motion with Monte Carlo method is one of the technique for calculating results. This paper presenting mathematical techniques what are using in financial mathematics for predicting future values in discrete domain. Applied on forecasting values of European Style Option prices, this was powerful tool in the ages behind us, when precise calculating of future values was very important. Whole financial industry, known as Quant Finance was developed in the period from 1960 up to today.*

Keywords: *Brownian motion, Forecasting Option Price, Monte Carlo Method*

INTRODUCTION

When on September 15, 2008 employees of Lehman Brothers started to flooding Times Square in New York, everybody in financial world in the USA saw some signs of crisis. But nobody expect that 10,000 employees will left building in 1585 Broadway, New York with the boxes with their personal things. Lehman Brothers were just the first in the queue of financial institutions who finished their business in USA financial market[1].

Not just Lehman Brothers, but top USA banks and financial institutions for more than four decades running business known as Hedge Funds. Behind that business was serious mathematics and statistics. Scientist known as a quants were specialists in programming, statistics, physics. The concept generally incorporates combinations of the following:

- forecasting value of the shares in the future
- buying and selling shares (options)
- monuitoring competition

But, before we start let me introduce term European Style Options. In finance, the *style or family* of an option is a general term denoting the class into which the option falls, usually defined by the dates on which the option may be exercised. The vast majority of options are either European or American (style) options.

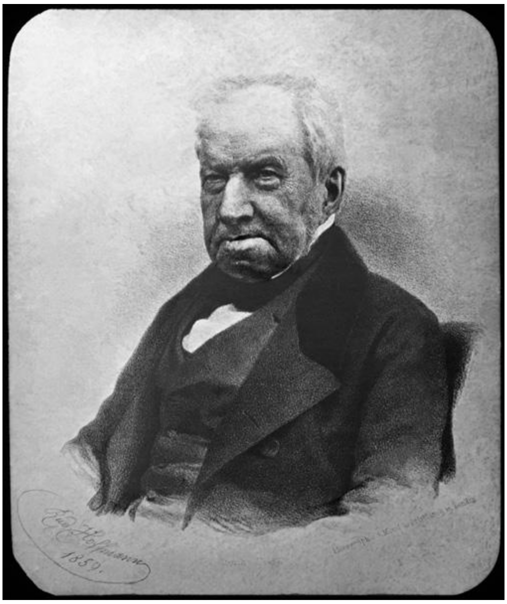
What is an Option?

The idea of options is certainly not new. Ancient Romans, Grecians, and Phoenicians traded options against outgoing cargoes from their local seaports. When used in relation to financial instruments, options are generally defined as a "contract between two parties in which one party has the right but not the obligation to do something, usually to buy or sell some underlying asset". Having rights without obligations has financial value, so option holders must purchase these rights, making them assets. This asset derives their value from some other asset, so they are called derivative assets. Call options are contracts giving the option holder the right to buy something, while put options, conversely, entitle the holder to sell something. Payment for call and put options, takes the form of a flat, up-front sum called a premium. Options can also be associated with bonds (i.e. convertible bonds and callable bonds), where payment occurs in installments over the entire life of the bond, but this paper is only concerned with traditional put and call options.

Origins of Option Pricing Techniques

Modern option pricing techniques, with roots in stochastic calculus, are often considered among the most mathematically complex of all applied areas of finance. These modern techniques derive their impetus from a formal history dating back to 1877, when Charles Castelli wrote a book entitled The Theory of Options in Stocks and Shares. Castelli's book introduced the public to the hedging and speculation aspects of options, but lacked any monumental theoretical base. Twenty three years later, Louis Bachelier offered the earliest known analytical valuation for options in his mathematics dissertation "Theorie de la Speculation" at the Sorbonne. He was on the right track, but he used a process to generate share price that allowed both negative security prices and option prices that exceeded the price of the underlying asset. Bachelier's work interested a professor at MIT named Paul Samuelson, who in 1955, wrote an unpublished paper entitled "Brownian Motion in the Stock Market". During that same year, Richard Kruizenga, one of Samuelson's students, cited Bachelier's work in his dissertation entitled "Put and Call Options: A Theoretical and Market Analysis". In 1962, another dissertation, this time by A. James Boness, focused on options. In his work, entitled "A Theory and Measurement of Stock Option Value", Boness developed a pricing model that made a significant theoretical jump from that of his predecessors. More significantly, his work served as a precursor to that of Fischer Black and Myron Scholes, who in 1973 introduced their landmark option pricing model.

BROWNIAN MOTION



Brownian motion (named after Robert Brown, who first observed the motion in 1827, when he examined pollen grains in water [2]), or pedesis (from Greek: πήδησις "leaping") is the assumably random movement of particles suspended in a fluid (i.e. a liquid such as water or a gas such as air) or the mathematical model used to describe such random movements, often called a particle theory. Brownian motion deals with the movement of solids from an area of high concentration to low concentration over a selectively permeable membrane.

The mathematical model of Brownian motion has several real-world applications. An often quoted example is stock market fluctuations. However, movements in share prices may arise due to unforeseen events which do not repeat themselves.

Brownian motion is among the simplest of the continuous-time stochastic (or probabilistic) processes, and it is a limit of both simpler and more complicated stochastic processes (see random walk and Donsker's theorem). This universality is closely related to the universality of the normal distribution. In both cases, it is often mathematical convenience rather than the accuracy of the models that motivates their use. This is because Brownian motion, whose time derivative is everywhere infinite, is an idealized approximation to actual random physical processes, which always have a finite time scale.

After Brownian works

However, it was Albert Einstein (in one of his 1905 papers) and Marian Smoluchowski (1906) who independently brought the solution of the problem to the attention of physicists, and presented it as a way to indirectly confirm the existence of atoms and molecules. Specifically, Einstein predicted that Brownian motion of a particle in a fluid at a thermodynamic temperature T is characterized by a diffusion coefficient

$D = k_B T / b$ (1)

Where:

-
$_B$ is Boltzmann's constant
-
is the linear drag coefficient on the particle (in the Stokes/low-Reynolds regime applicable for small particles).

As a consequence, the root mean square displacement in any direction after a time *t* is

$\sqrt{2Dt}$ (2)

At first the predictions of Einstein's formula were seemingly refuted by a series of experiments by Svedberg in 1906 and 1907, which gave displacements of the particles as 4 to 6 times the predicted value, and by Henri in 1908 who found displacements 3 times greater than Einstein's formula predicted. But Einstein's predictions were finally confirmed in a series of experiments carried out by Chaidesaigues in 1908 and Perrin in 1909. The confirmation of Einstein's theory constituted empirical progress for the kinetic theory of heat. In essence, Einstein showed that the motion can be predicted directly from the kinetic model of thermal equilibrium. The importance of the theory lay in the fact that it confirmed the kinetic theory's account of the second law of thermodynamics as being an essentially statistical law.

GEOMETRIC BROWNIAN MOTION MODEL

Geometric Brownian Motion time series are the most simple and commonly used for modeling in finance. Consider the formula:

$$x_{t+1} = x_t + Normal\ (\mu, \sigma) \qquad (3)$$

It says that the variable's value changes in one unit of time by an amount that is Normally distributed with mean μ and variance $\sigma 2$. The Normal distribution is a good first choice for a lot of variables because we can think of the model as saying (from Central Limit Theorem) that the variable x is being affected additively by many independent random variables. We can iterate the equation to give us the relationship between x_t and x_{t+2}:

$$x_{t+2} = x_{t+1} + Normal\ (\mu, \sigma) = x_t + Normal\ (\mu, \sigma) + Normal\ (\mu, \sigma) = x_t + Normal\ \left(2\mu, \sqrt{2}\sigma\right) \qquad (4)$$

and generalise to any time interval T:

$$x_{t+t} = x_t + Normal\ \left(\mu t, \sigma\sqrt{T}\right) \qquad (5)$$

This is a rather convenient equation because

a) we keep using Normal distributions, and

b) we can make a predictions between any time intervals we choose.

The above equation deals with discrete units of time but can be written in a continuous time form, where we consider any small time interval Δt:

$$\Delta x \quad = Normal\left(\mu\Delta t, \sigma\sqrt{\Delta T}\right) \quad (6)$$

The Stochastic Differential Equation (SDE) equivalent is:

$$dx = \mu dt + \sigma dz$$

$$dz = \varepsilon\sqrt{d_t} \qquad (7)$$

where *dz* is called a generalised Wiener process called variously the 'perturbation', innovation', or 'error', and e is a Normal(0,1) distribution. The notation might seem to be a rather unnecessary complication, but when you get used to the SDEs they give us the most succinct description of a stochastic time series. A more general version of the above equations is [5]:

$$dx = g(t)\, dt + f(t) dz$$

$$dz = \varepsilon dt \quad (8)$$

where *g* and *f* are two functions. It is really just shorthand for writing:

$$x(t) = \int_0^t g(\tau) d\tau + \int_0^t f(\tau)\, dz(\tau) \quad (9)$$

The equation $x_{t+1} = x_t Normal\ (\mu, \sigma)$ allows the variable *x* to take any real value, including negative values, so it would not be much good at modelling a stock price, interest rate or exchange rate for example. However, it has the desirable property of being memory less, i.e. to make a prediction of the value of *x* some time *T* from now we only need to know the value of *x* now, not anything about the path it took to get to the present value. We can model the return of a stock:

$$\frac{dS}{S} = r = \mu dt + \sigma dz \quad (10)$$

or

$$dS = \mu S dt + \sigma S dz \quad (11)$$

There is an identity known as *Itô's lemma* which says that for a function *F* of a stochastic variable *X*:

$$dF = \frac{dF}{dX} dX + \frac{1}{2} \frac{d^2F}{dX^2} dt \quad (12)$$

Since *dS/S = d(log[S])* we can rewrite, using *F(S) = log[S]:*

$$\frac{dS}{S} = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz \quad (13)$$

Integrating over time T we get the relationship between some initial value S_t and some later value S_{t+T}:

$$S_{t+T} = S_T \exp\left[Normal\ \left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)\right] = S_T \exp[r_T] \quad (14)$$

where r_T is the return of the stock over the period *T*. The *Exp[...]* term in this equation means that *S* is always >0 so we still retain the memoryless property which corresponds to some financial thinking that a stock's value encompasses all information available about a stock at the time so there should be no memory in the system.

The return *r* of a stock *S* is the log of the fractional change in the stock's value. For stocks this is a more interesting value than the stock's actual price because it would be more profitable to own 10 shares in a $1 stock that increased by 6% over a year than 1 share in a $10 stock that increased by 4%, for example.

This last equation is what we call the GBM model: 'the 'geometric' part comes because we are effectively multiplying lots of distributions together (adding them in log space). From the definition of a Lognormal random variable, if *ln[S]* is Normally distributed then *S* is Lognormally distributed, so Equation for *St+T* is modelling it as a Lognormal random variable. From the Lognormal E equations you can see that *St+T* has a mean given by:

$$E(S_{t+T}) = S_t \exp[\mu T] \quad (15)$$

hence μ is also called the exponential growth rate, and a variance given by:

$$V(S_{t+T}) = exp[2\mu T](exp[\sigma^2 T] - 1) \quad (16)$$

The spread of possible values in a GBM increases rapidly with time. For example, the following plot shows 50 possible forecasts with S_0 = 1, m = 0.001 and s = 0.02:

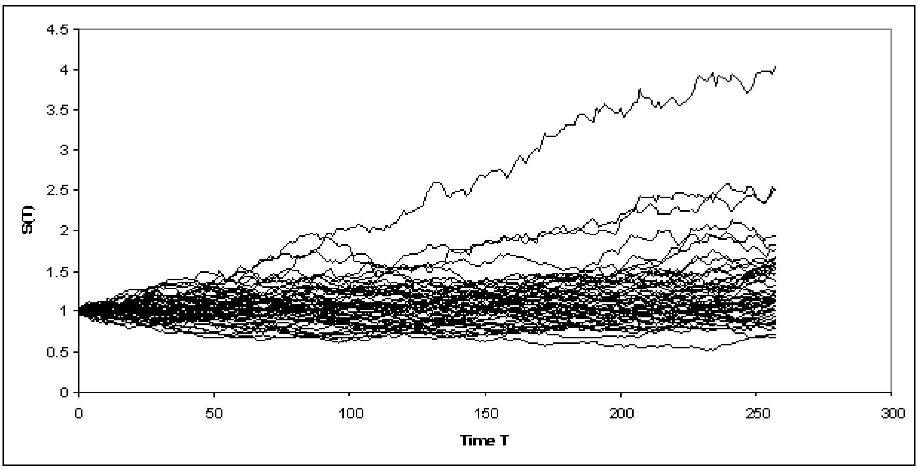


Figure 2. 50 possible forecasts with S_0 = 1, m = 0.001 and s = 0.02

CONCLUSION

In financial forecast we using following simple formula, what is derived from equation 16:

$$S_t \sim S_0 \exp\left([\mu - \tfrac{1}{2}\sigma^2]t + \sigma\sqrt{t}N_{0,1}\right) \quad (17)$$

S_0 - Initial value at t=0 of geometric Brownian

S_t -Value of geometric Brownian motion at time t

μ −Drift term

σVolatility

$N_{0,1}$ - Random sample from a normal (Gaussian) distribution with mean 0 and standard deviation 1

Using formula (17) we can calculate discrete values for Brownian motion. In addition with Monte Carlo Method we easily can generate values for certain amount of days. This formula is good tool for calculating options and shares in stock exchange, as well as future values of random series.

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